

GENERALIZED FRACTIONAL MAXIMAL FUNCTIONS IN LORENTZ SPACES

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ABSTRACT. In this paper we give the complete characterization of the boundedness of the generalized fractional maximal operator

$$M_{\phi, \Lambda^\alpha(b)} f(x) := \sup_{Q \ni x} \frac{\|f \chi_Q\|_{\Lambda^\alpha(b)}}{\phi(|Q|)} \quad (x \in \mathbb{R}^n),$$

between the classical Lorentz spaces $\Lambda^p(v)$ and $\Lambda^q(w)$ for appropriate functions ϕ , where $0 < p, q < \infty$, $0 < \alpha \leq r < \infty$, v, w, b are weight functions on $(0, \infty)$ such that $0 < B(x) := \int_0^x b < \infty$, $x > 0$, $B \in \Delta_2$ and $B(t)/t^{\alpha/r}$ is quasi-increasing.

1. INTRODUCTION

Throughout the paper, we always denote by c or C a positive constant, which is independent of main parameters but it may vary from line to line. However a constant with subscript such as c_1 does not change in different occurrences. By $a \lesssim b$, we mean that $a \leq \lambda b$, where $\lambda > 0$ depends on inessential parameters. If $a \lesssim b$ and $b \lesssim a$, we write $a \approx b$ and say that a and b are equivalent. Unless a special remark is made, the differential element dx is omitted when the integrals under consideration are the Lebesgue integrals. By a cube, we mean an open cube with sides parallel to the coordinate axes.

Given two quasi-normed vector spaces X and Y , we write $X = Y$ if X and Y are equal in the algebraic and the topological sense (their quasi-norms are equivalent). The symbol $X \hookrightarrow Y$ ($Y \hookleftarrow X$) means that $X \subset Y$ and the natural embedding I of X in Y is continuous, that is, there exist a constant $c > 0$ such that $\|z\|_Y \leq c\|z\|_X$ for all $z \in X$. The best constant of the embedding $X \hookrightarrow Y$ is $\|I\|_{X \rightarrow Y}$.

Let Ω be any measurable subset of \mathbb{R}^n , $n \geq 1$. Let $\mathfrak{M}(\Omega)$ denote the set of all measurable functions on Ω and $\mathfrak{M}_0(\Omega)$ the class of functions in $\mathfrak{M}(\Omega)$ that are finite a.e. The symbol $\mathfrak{M}^+(\Omega)$ stands for the collection of all $f \in \mathfrak{M}(\Omega)$ which are non-negative on Ω . The symbol $\mathfrak{M}^+((0, \infty); \downarrow)$ is used to denote the subset of those functions from $\mathfrak{M}^+(0, \infty)$ which are non-increasing on $(0, \infty)$. Denote by $\mathfrak{M}^{\text{rad}, \downarrow} = \mathfrak{M}^{\text{rad}, \downarrow}(\mathbb{R}^n)$ the set of all measurable, non-negative, radially decreasing functions on \mathbb{R}^n , that is,

$$\mathfrak{M}^{\text{rad}, \downarrow} := \{f \in \mathfrak{M}(\mathbb{R}^n) : f(x) = h(|x|), x \in \mathbb{R}^n \text{ with } h \in \mathfrak{M}^+((0, \infty); \downarrow)\}.$$

The family of all weight functions (also called just weights) on Ω , that is, locally integrable non-negative functions on Ω , is given by $\mathcal{W}(\Omega)$. Everywhere in the paper, u , v and w are weights.

For $p \in (0, \infty]$ and $w \in \mathfrak{M}^+(\Omega)$, we define the functional $\|\cdot\|_{p, w, \Omega}$ on $\mathfrak{M}(\Omega)$ by

$$\|f\|_{p, w, \Omega} := \begin{cases} \left(\int_{\Omega} |f(x)|^p w(x) dx \right)^{1/p} & \text{if } p < \infty, \\ \text{ess sup}_{\Omega} |f(x)| w(x) & \text{if } p = \infty. \end{cases}$$

If, in addition, $w \in \mathcal{W}(\Omega)$, then the weighted Lebesgue space $L^p(w, \Omega)$ is given by

$$L^p(w, \Omega) = \{f \in \mathfrak{M}(\Omega) : \|f\|_{p, w, \Omega} < \infty\}$$

and it is equipped with the quasi-norm $\|\cdot\|_{p, w, \Omega}$.

When $w \equiv 1$ on Ω , we write simply $L^p(\Omega)$ and $\|\cdot\|_{p, \Omega}$ instead of $L^p(w, \Omega)$ and $\|\cdot\|_{p, w, \Omega}$, respectively.

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Suppose f is a measurable a.e. finite function on \mathbb{R}^n . Then its non-increasing rearrangement f^* is given by

$$f^*(t) = \inf\{\lambda > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| \leq t\}, \quad t \in (0, \infty),$$

and let f^{**} denotes the Hardy-Littlewood maximal function of f , i.e.

$$f^{**}(t) := \frac{1}{t} \int_0^t f^*(\tau) d\tau, \quad t > 0.$$

Quite many familiar function spaces can be defined using the non-increasing rearrangement of a function. One of the most important classes of such spaces are the so-called classical Lorentz spaces.

Let $p \in (0, \infty)$ and $w \in \mathcal{W}(0, \infty)$. Then the classical Lorentz spaces $\Lambda^p(w)$ and $\Gamma^p(w)$ consist of all functions $f \in \mathfrak{M}(\mathbb{R}^n)$ for which $\|f\|_{\Lambda^p(w)} := \|f^*\|_{p,w,(0,\infty)} < \infty$ and $\|f\|_{\Gamma^p(w)} := \|f^{**}\|_{p,w,(0,\infty)} < \infty$, respectively. For more information about the Lorentz Λ and Γ see e.g. [7] and the references therein.

A weak-type modification of the space $\Lambda^p(w)$ is defined by (cf. [9, 62])

$$\Lambda^{p,\infty}(w) := \left\{ f \in \mathfrak{M}(\mathbb{R}^n) : \|f\|_{\Lambda^{p,\infty}(w)} := \sup_{t>0} f^*(t) \left(\int_0^t w(\tau) d\tau \right)^{1/p} < \infty \right\}.$$

One can easily see that $\Lambda^p(w) \hookrightarrow \Lambda^{p,\infty}(w)$. Recall that classical and weak-type Lorentz spaces include many familiar spaces (see, for instance, [16]).

The study of maximal operators is one of the most important topics in harmonic analysis. These significant non-linear operators, whose behavior are very informative in particular in differentiation theory, provided the understanding and the inspiration for the development of the general class of singular and potential operators (see, for instance, [19, 35–37, 66–68]).

The main example is the Hardy-Littlewood maximal function which is defined for locally integrable functions f on \mathbb{R}^n by

$$Mf(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy = \frac{\|f\chi_Q\|_{1,\mathbb{R}^n}}{\|\chi_Q\|_{1,\mathbb{R}^n}}, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes Q containing x .

On using the Herz and the Stein rearrangement inequalities

$$(1.1) \quad cf^{**}(t) \leq (Mf)^*(t) \leq Cf^{**}(t), \quad t \in (0, \infty),$$

where c and C are positive constants depending only on n (cf. [4, Chapter 3, Theorem 3.8]), it is clear that in order to describe mapping properties of the Hardy-Littlewood maximal operator between the classical Lorentz space $\Lambda^p(v)$ and $\Lambda^q(w)$, one has to characterize the weights v, w for which the inequality

$$(1.2) \quad \left(\int_0^\infty \left(\int_0^t f^*(\tau) d\tau \right)^q w(t) dt \right)^{1/q} \lesssim \left(\int_0^\infty f^*(t)^p v(t) dt \right)^{1/p}$$

holds. The first results on the problem $\Lambda^p(v) \hookrightarrow \Gamma^p(v)$, $1 < p < \infty$, which is equivalent to inequality (1.2), were obtained by Boyd [5] and in an explicit form by Ariño and Muckenhoupt [1]. The problem with $w \neq v$ and $p \neq q$, $1 < p, q < \infty$ was first successfully solved by Sawyer [59]. Many articles on this topic followed, providing the results for a wider range of parameters. In particular, much attention was paid to inequality (1.2); see for instance [1, 3, 6, 10, 20–25, 31–34, 38, 39, 44, 57, 59–61, 63, 64], survey [7], the monographs [42, 43], for the latest development of this subject see [27, 31], and references given there.

The fractional maximal operator, M_γ , $\gamma \in (0, n)$, is defined at $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ by

$$(M_\gamma f)(x) := \sup_{Q \ni x} |Q|^{\gamma/n-1} \int_Q |f(y)| dy, \quad x \in \mathbb{R}^n.$$

It was shown in [12, Theorem 1.1] that

$$(1.3) \quad (M_\gamma f)^*(t) \lesssim \sup_{\tau > t} \tau^{\gamma/n-1} \int_0^\tau f^*(y) dy \lesssim (M_\gamma \tilde{f})^*(t)$$

for every $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $t \in (0, \infty)$, where $\tilde{f}(x) := f^*(\omega_n |x|^n)$ and ω_n is the volume of S^{n-1} . Thus, in order to characterize boundedness of the fractional maximal operator M_γ between classical Lorentz spaces $\Lambda^p(v)$ and $\Lambda^q(w)$ it is necessary and sufficient to characterize the validity of the weighted inequality

$$(1.4) \quad \left(\int_0^\infty \left[\sup_{\tau > t} \tau^{\gamma/n-1} \int_0^\tau \phi(y) dy \right]^q w(t) dt \right)^{1/q} \lesssim \left(\int_0^\infty [\phi(t)]^p v(t) dt \right)^{1/p}$$

for all $\phi \in \mathfrak{M}^+((0, \infty); \downarrow)$. Such a characterization was obtained in [12] for the particular case when $1 < p \leq q < \infty$ and in [50, Theorem 2.10] in the case of more general operators and for extended range of p and q .

Let $s \in (0, \infty)$, $\gamma \in [0, n)$ and $\mathbb{A} = (A_0, A_\infty) \in \mathbb{R}^2$. Denote by

$$\ell^{\mathbb{A}}(t) := (1 + |\log t|)^{A_0} \chi_{[0,1]}(t) + (1 + |\log t|)^{A_\infty} \chi_{[1,\infty)}(t), \quad (t > 0).$$

Recall that the fractional maximal operator $M_{s,\gamma,\mathbb{A}}$ at $f \in \mathfrak{M}(\mathbb{R}^n)$ defined in [16] by

$$(M_{s,\gamma,\mathbb{A}}f)(x) := \sup_{Q \ni x} \frac{\|f \chi_Q\|_s}{\|\chi_Q\|_{sn/(n-\gamma),\mathbb{A}}}, \quad x \in \mathbb{R}^n$$

satisfies the following equivalency

$$(M_{s,\gamma,\mathbb{A}}f)(x) \approx \sup_{Q \ni x} \frac{\|f \chi_Q\|_s}{|Q|^{(n-\gamma)/(sn)} \ell^{\mathbb{A}}(|Q|)}, \quad x \in \mathbb{R}^n.$$

Hence, if $s = 1$, $\gamma = 0$ and $\mathbb{A} = (0, 0)$, then $M_{s,\gamma,\mathbb{A}}$ is equivalent to the classical Hardy-Littlewood maximal operator M . If $s = 1$, $\gamma \in (0, n)$ and $\mathbb{A} = (0, 0)$, then $M_{s,\gamma,\mathbb{A}}$ is equivalent to the usual fractional maximal operator M_γ . Moreover, if $s = 1$, $\gamma \in [0, n)$ and $\mathbb{A} \in \mathbb{R}^2$, then $M_{s,\gamma,\mathbb{A}}$ is the fractional maximal operator which corresponds to potentials with logarithmic smoothness treated in [52, 53]. In particular, if $\gamma = 0$, then $M_{1,\gamma,\mathbb{A}}$ is the maximal operator of purely logarithmic order.

It was shown in [16, Theorem 3.1] that if $s \in (0, \infty)$, $\gamma \in [0, n)$ and $\mathbb{A} = (A_0, A_\infty) \in \mathbb{R}^2$ satisfy either $\gamma \in (0, n)$, or $\gamma = 0$ and $A_0 \geq 0 \geq A_\infty$, then there exists a constant $C > 0$ depending only in n, s, γ and \mathbb{A} such that for all $f \in \mathfrak{M}(\mathbb{R}^n)$ and every $t \in (0, \infty)$

$$(1.5) \quad (M_{s,\gamma,\mathbb{A}}f)^*(t) \leq C \left[\sup_{t \leq \tau < \infty} \tau^{\gamma/n-1} \ell^{-s\mathbb{A}}(\tau) \int_0^\tau (f^*)^s(y) dy \right]^{1/s}.$$

Inequality (1.5) is sharp in the sense that for every $\varphi \in \mathfrak{M}^+((0, \infty); \downarrow)$ there exists a function $f \in \mathfrak{M}(\mathbb{R}^n)$ such that $f^* = \varphi$ a.e. on $(0, \infty)$ and for all $t \in (0, \infty)$,

$$(M_{s,\gamma,\mathbb{A}}f)^*(t) \geq c \left[\sup_{\tau > t} \tau^{\gamma/n-1} \ell^{-s\mathbb{A}}(\tau) \int_0^\tau (f^*)^s(y) dy \right]^{1/s},$$

where c is a positive constant with again depends only on n, s, γ and \mathbb{A} . Consequently, the operator $M_{s,\gamma,\mathbb{A}} : \Lambda^p(v) \rightarrow \Lambda^q(w)$ is bounded if and only if the inequality

$$(1.6) \quad \left(\int_0^\infty \left[\sup_{\tau > t} \tau^{\gamma/n-1} \ell^{-s\mathbb{A}}(\tau) \int_0^\tau \varphi(y) dy \right]^{q/s} w(t) dt \right)^{s/q} \lesssim \left(\int_0^\infty \varphi^{p/s}(t) v(t) dt \right)^{s/p}$$

holds for all $\phi \in \mathfrak{M}^+((0, \infty); \downarrow)$. The complete characterization of inequality (1.6) was given in [16, p. 17 and p. 34]. Full proofs and some further extensions and applications can be found in [16], [17].

Given p and q , $0 < p, q < \infty$, let $M_{p,q}$ denote the maximal operator associated to the Lorentz $L^{p,q}$ spaces defined by

$$M_{p,q}f(x) := \sup_{Q \ni x} \frac{\|f \chi_Q\|_{p,q}}{\|\chi_Q\|_{p,q}} = \sup_{Q \ni x} \frac{\|f \chi_Q\|_{p,q}}{|Q|^{1/p}},$$

where $\|\cdot\|_{p,q}$ is the usual Lorentz norm

$$\|f\|_{p,q} := \left(\int_0^\infty [\tau^{1/p} f^*(\tau)]^q \frac{d\tau}{\tau} \right)^{1/q}.$$

This operator was introduced by Stein in [65] in order to obtain certain endpoint results in differentiation theory. The operator $M_{p,q}$ have been also considered by other authors, for instance see [2, 45, 46, 49, 55].

It was proved in [2], with the help of interpolation, that if $1 \leq q \leq p$, then

$$(1.7) \quad (M_{p,q}f)^*(t) \lesssim \frac{1}{t^{1/p}} \left(\int_0^t f^*(\tau)^q \tau^{q/p-1} d\tau \right)^{1/q}.$$

This result was extended to more general setting of maximal operators in [48]. Consequently, if one knows the characterization of the weights u, v, w for which the inequality

$$(1.8) \quad \left(\int_0^\infty \left(\int_0^t f^*(\tau) u(\tau) d\tau \right)^\beta w(t) dt \right)^{1/\beta} \lesssim \left(\int_0^\infty f^*(t)^\alpha v(t) dt \right)^{1/\alpha}$$

holds, then it is possible to describe mapping properties of $M_{p,q}$ between the classical Lorentz spaces $\Lambda^\alpha(v)$ and $\Lambda^\beta(w)$ when $1 \leq q \leq p$.

Let $u \in \mathcal{W}(0, \infty) \cap C(0, \infty)$, $b \in \mathcal{W}(0, \infty)$ and $B(t) := \int_0^t b(s) ds$. Assume that b is such that $0 < B(t) < \infty$ for every $t \in (0, \infty)$. The iterated Hardy-type operator involving suprema $T_{u,b}$ is defined at $g \in \mathfrak{M}^+(0, \infty)$ by

$$(T_{u,b}g)(t) := \sup_{t \leq \tau < \infty} \frac{u(\tau)}{B(\tau)} \int_0^\tau g(y) b(y) dy, \quad t \in (0, \infty).$$

It is easy to see that the left-hand sides of inequalities (1.2), (1.4), (1.6) and (1.8) can be interpreted as a particular examples of operators $T_{u,b}$.

Such operators have been found indispensable in the search for optimal pairs of rearrangement-invariant norms for which a Sobolev-type inequality holds (cf. [41]). They constitute a very useful tool for characterization of the associate norm of an operator-induced norm, which naturally appears as an optimal domain norm in a Sobolev embedding (cf. [54], [56]). Supremum operators are also very useful in limiting interpolation theory as can be seen from their appearance for example in [18], [15], [14], [58].

In [28], complete characterization for the inequality

$$(1.9) \quad \|T_{u,b}f\|_{q,w,(0,\infty)} \leq c \|f\|_{p,v,(0,\infty)}, \quad f \in \mathfrak{M}^\downarrow(0, \infty)$$

for $0 < q < \infty$, $0 < p < \infty$ is given (see Theorem 2.6).

Inequality (1.9) was characterized in [29, Theorem 3.5] under additional condition

$$\sup_{0 < t < \infty} \frac{u(t)}{B(t)} \int_0^t \frac{b(\tau)}{u(\tau)} d\tau < \infty.$$

Note that the case when $0 < p \leq 1 < q < \infty$ was not considered in [29]. It is also worth to mention that in the case when $1 < p < \infty$, $0 < q < p < \infty$, $q \neq 1$ [29, Theorem 3.5] contains only discrete condition. In [22] the new reduction theorem was obtained when $0 < p \leq 1$, and this technique allowed to characterize inequality (1.9) when $b \equiv 1$, and in the case when $0 < q < p \leq 1$, [22] contains only discrete condition.

In this paper we define the following generalized fractional maximal operator $M_{\phi, \Lambda^\alpha(b)}$ and characterize the boundedness of this operator between classical and weak-type Lorentz spaces by reducing the problem to the boundedness of the operator $T_{u,b}$ in weighted Lebesgue spaces on the cone of non-negative non-increasing functions:

Let $0 < \alpha < \infty$, $b \in \mathcal{W}$ and $\phi : (0, \infty) \rightarrow (0, \infty)$. Denote by

$$(1.10) \quad M_{\phi, \Lambda^\alpha(b)} f(x) := \sup_{Q \ni x} \frac{\|f \chi_Q\|_{\Lambda^\alpha(b)}}{\phi(|Q|)} \quad (x \in \mathbb{R}^n).$$

Note that $M_{\phi, \Lambda^\alpha(b)} = M_\gamma$, where M_γ is the fractional maximal operator, when $\alpha = 1$, $b \equiv 1$ and $\phi(t) = t^{1-\gamma/n}$ ($t > 0$) with $0 < \gamma < n$. Moreover, $M_{\phi, \Lambda^\alpha(b)} \approx M_{s, \gamma, \mathbb{A}}$, when $\alpha = s$, $b \equiv 1$ and $\phi(t) = t^{(n-\gamma)/(sn)} \ell^\mathbb{A}(t)$, ($t > 0$) with $0 < \gamma < n$ and $\mathbb{A} = (A_0, A_\infty) \in \mathbb{R}^2$. It is worth also to mention that $M_{\phi, \Lambda^\alpha(b)} = M_{p, q}$, when $\alpha = q$, $b(t) = t^{q/p-1}$ and $\phi(t) = t^{1/p}$ ($t > 0$).

The paper is organized as follows. Section 2 contains some preliminaries along with the standard ingredients used in the proofs. The main results are stated and proved in Section 3.

2. NOTATIONS AND PRELIMINARIES

Let F be any non-negative set function defined on the collection of all sets of positive finite measure. Define its maximal function by

$$MF(x) := \sup_{Q \ni x} F(Q),$$

where the supremum is taken over all cubes containing x .

Definition 2.1. [46, Definition 1] We say that a set function F is pseudo-increasing if there is a positive constant $C > 0$ such that for any finite collection of pairwise disjoint cubes $\{Q_j\}$, we have

$$(2.1) \quad \min_i F(Q_i) \leq CF\left(\bigcup_i Q_i\right).$$

Theorem 2.2. [46, Theorem 1] Let F be a pseudo-increasing set function. Then, for any $t > 0$,

$$(2.2) \quad (MF)^*(t) \leq C \sup_{|E| > t/3^n} F(E),$$

where C is the constant appearing in (2.1), and the supremum is taken over all sets E of finite measure $|E| > t/3^n$.

We will need the following elementary inequality [4, p. 44]

$$(2.3) \quad \int_E |f(x)| dx \leq \int_0^{|E|} f^*(\xi) d\xi.$$

A function $\phi : (0, \infty) \rightarrow (0, \infty)$ is said to satisfy the Δ_2 -condition, denoted $\phi \in \Delta_2$, if for some $C > 0$

$$\phi(2t) \leq C\phi(t) \quad \text{for all } t > 0.$$

A function $\phi : (0, \infty) \rightarrow (0, \infty)$ is said to be quasi-increasing (quasi-decreasing), if for some $C > 0$

$$\phi(t_1) \leq C\phi(t_2) \quad (\phi(t_2) \leq c\phi(t_1)),$$

whenever $0 < t_1 \leq t_2 < \infty$.

A function $\phi : (0, \infty) \rightarrow (0, \infty)$ is said to satisfy the Q_r -condition, $0 < r < \infty$, denoted $\phi \in Q_r$, if for some constant $C > 0$

$$\phi\left(\sum_{i=1}^n t_i\right) \leq C\left(\sum_{i=1}^n \phi(t_i)^r\right)^{1/r},$$

for every finite set of non-negative real numbers $\{t_1, \dots, t_n\}$.

It is clear that if $\omega \in Q_r$, $0 < r < \infty$ and $g : (0, \infty) \rightarrow (0, \infty)$ is a quasi-decreasing function, then $\omega \cdot g \in Q_r$.

A quasi-Banach space X is a complete metrizable real vector space whose topology is given by a quasi-norm $\|\cdot\|$ satisfying the following three conditions: $\|x\| > 0$, $x \in X$, $x \neq 0$; $\|\lambda x\| = |\lambda| \|x\|$, $\lambda \in \mathbb{R}$, $x \in X$; and $\|x_1 + x_2\| \leq C(\|x_1\| + \|x_2\|)$, $x_1, x_2 \in X$, where $C \geq 1$ is a constant independent of x_1 and x_2 .

A quasi-Banach function space on a measure space (\mathbb{R}^n, dx) is defined to be a quasi-Banach space X which is a subspace of $\mathfrak{M}_0(\mathbb{R}^n)$ (the topological linear space of all equivalence classes of the real Lebesgue measurable functions equipped with the topology of convergence in measure) such that there exists $u \in X$ with $u > 0$ a.e. and if $|f| \leq |g|$ a.e., where $g \in X$ and $f \in \mathfrak{M}_0(\mathbb{R}^n)$, then $f \in X$ and $\|f\|_X \leq \|g\|_X$.

A quasi-Banach function space X is said to be order continuous if for every $f \in X$ and every sequence $\{f_n\}$ such that $0 \leq f_n \leq |f|$ and $f_n \downarrow 0$ a.e. it holds $\|f_n\|_X \rightarrow 0$.

A quasi-Banach function space X is said to satisfy a lower r -estimate, $0 < r < \infty$, if there exists a constant C such that

$$\left(\sum_{i=1}^n \|f_i\|_X^r \right)^{1/r} \leq C \left\| \sum_{i=1}^n f_i \right\|_X,$$

for every finite set of functions $\{f_1, \dots, f_n\} \subset X$ with pairwise disjoint supports (see [47, 1.f.4]).

A quasi-Banach function space $(X, \|\cdot\|_X)$ of real-valued, locally integrable, Lebesgue measurable functions on \mathbb{R}^n is said to be a rearrangement-invariant (r.i.) space if it satisfies the following conditions:

- (1) If $g^* \leq f^*$ and $f \in X$, then $g \in X$ with $\|g\|_X \leq \|f\|_X$.
- (2) If A is a Lebesgue measurable set of finite measure, then $\chi_A \in X$.
- (3) $0 \leq f_n \uparrow$, $\sup_{n \in \mathbb{N}} \|f_n\|_X \leq M$, imply that $f = \sup_{n \in \mathbb{N}} f_n \in X$ and $\|f\|_X = \sup_{n \in \mathbb{N}} \|f_n\|_X$.

For each r.i. space X on \mathbb{R}^n , a r.i. space \bar{X} on $(0, +\infty)$ is associated such that $f \in X$ if and only if $f^* \in \bar{X}$ and $\|f\|_X = \|f^*\|_{\bar{X}}$ (see [4]).

Most of the properties of r.i. spaces can be formulated in terms of the fundamental function φ_X of X defined by

$$\varphi_X(t) = \|\chi_E\|_X,$$

where $|E| = t$. Observe that the particular choice of the set E with $|E| = t$ is immaterial by the rearrangement-invariance of X . The function φ_X is quasi-concave and continuous, except perhaps at the origin.

Let X be a quasi-Banach function space on \mathbb{R}^n . By X_{loc} we denote the space of all $f \in \mathfrak{M}_0(\mathbb{R}^n)$ such that $f\chi_Q \in X$ for every cube $Q \subset \mathbb{R}^n$.

Definition 2.3. Suppose that X is a quasi-Banach space of measurable functions defined on \mathbb{R}^n . Given a function $\phi : (0, \infty) \rightarrow (0, \infty)$, denote for every $f \in X_{\text{loc}}$ by

$$(2.4) \quad M_{\phi, X} f(x) := \sup_{Q \ni x} \frac{\|f\chi_Q\|_X}{\phi(|Q|)} \quad (x \in \mathbb{R}^n).$$

It is easy to see that $M_{\phi, X} f$ is a lower-semicontinuous function.

Note that if X is r.i. quasi-Banach function space on \mathbb{R}^n , then

$$M_{\varphi_X, X} f(x) = \sup_{Q \ni x} \frac{\|f\chi_Q\|_X}{\|\chi_Q\|_X} \quad (x \in \mathbb{R}^n).$$

It was shown in [2, Theorem 1], in particular, that if X is a r.i. quasi-Banach function space satisfying a lower φ_X -estimate, that is, if there exists $C > 0$ such that for all $m \in \mathbb{N}$ and $\{f_i\}_{i=1}^m \subset X$ with pairwise disjoint supports we have

$$\varphi_X \left(\sum_{i=1}^n \varphi_X^{-1}(\|f_i\|_X) \right) \leq C \left\| \sum_{i=1}^n f_i \right\|_X,$$

then there exists $C > 0$ such that for all $f \in X$ the inequality

$$\sup_{t>0} \varphi_X(t) (M_{\varphi_X, X})^*(t) \leq C \|f\|_X.$$

It was proved in [13, Theorem 3.5] that if X is a r.i. order continuous quasi-Banach function space satisfying a lower φ_X -estimate, then X has the Lebesgue differentiation property, that is,

$$\lim_{r \rightarrow 0} \frac{\|(f - f(x))\chi_{B(x,r)}\|_X}{\|\chi_{B(x,r)}\|_X} = 0$$

for all $f \in X$ and for a.a. $x \in \mathbb{R}^n$.

Denote by

$$V(x) := \int_0^x v(t) dt \text{ and } W(x) := \int_0^x w(t) dt \text{ for all } x > 0.$$

Suppose $0 < p < \infty$ and let w be a weight on $(0, \infty)$ such that $W \in \Delta_2$ and $W(\infty) = \infty$. Then the classical Lorentz space $\Lambda^p(w)$ is a r.i. order-continuous quasi-Banach function space (see, for instance, [8, Section 2.2] and [40]).

The following statement was proved in [40].

Theorem 2.4. [40, Theorem 7] *Let w be a weight function such that $W \in \Delta_2$. Given $0 < p, r < \infty$, the following assertions are equivalent:*

- (i) $\Lambda^p(w)$ satisfies a lower r -estimate.
- (ii) $W(t)/t^{p/r}$ is quasi-increasing and $r \geq p$.

We adopt the following conventions:

Convention 2.5. (i) Throughout the paper we put $0 \cdot \infty = 0$, $\infty/\infty = 0$ and $0/0 = 0$.

(ii) If $p \in [1, +\infty]$, we define p' by $1/p + 1/p' = 1$.

(iii) If $0 < q < p < \infty$, we define r by $1/r = 1/q - 1/p$.

Finally, for the convenience of the reader, we recall the above-mentioned characterization of the inequality (1.9), when $0 < p, q < \infty$.

Theorem 2.6. [28, Theorems 5.1 and 5.5] *Let $0 < p, q < \infty$ and let $u \in \mathcal{W}(0, \infty) \cap C(0, \infty)$. Assume that $b, v, w \in \mathcal{W}(0, \infty)$ is such that $0 < B(t) < \infty$, $0 < V(x) < \infty$ and $0 < W(x) < \infty$ for all $x > 0$. Then inequality (1.9) is satisfied with the best constant c if and only if:*

- (i) $1 < p \leq q$, and in this case $c \approx A_1 + A_2$, where

$$A_1 := \sup_{x>0} \left(\left[\sup_{x \leq \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q W(x) + \int_x^\infty \left[\sup_{t \leq \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q w(t) dt \right)^{1/q} \left(\int_0^x \left(\frac{B(y)}{V(y)} \right)^{p'} v(y) dy \right)^{1/p'},$$

$$A_2 := \sup_{x>0} \left(\left[\sup_{x \leq \tau < \infty} \frac{u(\tau)}{V^2(\tau)} \right]^q W(x) + \int_x^\infty \left[\sup_{t \leq \tau < \infty} \frac{u(\tau)}{V^2(\tau)} \right]^q w(t) dt \right)^{1/q} \left(\int_0^x V^{p'}(y) v(y) dy \right)^{1/p'};$$

- (ii) $1 = p \leq q$, and in this case $c \approx B_1 + B_2$, where

$$B_1 := \sup_{x>0} \left(\left[\sup_{x \leq \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q W(x) + \int_x^\infty \left[\sup_{t \leq \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q w(t) dt \right)^{1/q} \left(\sup_{0 < y \leq x} \frac{B(y)}{V(y)} \right),$$

$$B_2 := \sup_{x>0} \left(\left[\sup_{x \leq \tau < \infty} \frac{u(\tau)}{V^2(\tau)} \right]^q W(x) + \int_x^\infty \left[\sup_{t \leq \tau < \infty} \frac{u(\tau)}{V^2(\tau)} \right]^q w(t) dt \right)^{1/q} V(x);$$

- (iii) $1 < p$ and $q < p$, and in this case $c \approx C_1 + C_2 + C_3 + C_4$, where

$$C_1 := \left(\int_0^\infty \left(\int_x^\infty \left[\sup_{t \leq \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q w(t) dt \right)^{r/p} \left[\sup_{x \leq \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q \left(\int_0^x \left(\frac{B(y)}{V(y)} \right)^{p'} v(y) dy \right)^{r/p'} w(x) dx \right)^{1/r},$$

$$C_2 := \left(\int_0^\infty W^{r/p}(x) \left[\sup_{x \leq \tau < \infty} \left[\sup_{\tau \leq y < \infty} \frac{u(y)}{B(y)} \right] \left(\int_0^x \left(\frac{B(y)}{V(y)} \right)^{p'} v(y) dy \right)^{1/p'} \right]^r w(x) dx \right)^{1/r},$$

$$C_3 := \left(\int_0^\infty \left(\int_x^\infty \left[\sup_{t \leq \tau < \infty} \frac{u(\tau)}{V^2(\tau)} \right]^q w(t) dt \right)^{r/p} \left[\sup_{x \leq \tau < \infty} \frac{u(\tau)}{V^2(\tau)} \right]^q \left(\int_0^x V^{p'}(y) v(y) dy \right)^{r/p'} w(x) dx \right)^{1/r},$$

$$C_4 := \left(\int_0^\infty W^{r/p}(x) \left[\sup_{x \leq \tau < \infty} \left[\sup_{\tau \leq y < \infty} \frac{u(y)}{V^2(y)} \right] \left(\int_0^x V^{p'}(y) v(y) dy \right)^{1/p'} \right]^r w(x) dx \right)^{1/r};$$

(iv) $q < 1 = p$, and in this case $c \approx D_1 + D_2 + D_3 + D_4$, where

$$\begin{aligned} D_1 &:= \left(\int_0^\infty \left(\int_x^\infty \left[\sup_{t \leq \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q w(t) dt \right)^{r/p} \left[\sup_{x \leq \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q \left(\sup_{0 < y \leq x} \frac{B(y)}{V(y)} \right)^r w(x) dx \right)^{1/r}, \\ D_2 &:= \left(\int_0^\infty W^{r/p}(x) \left[\sup_{x \leq \tau < \infty} \left[\sup_{\tau \leq y < \infty} \frac{u(y)}{B(y)} \right] \left(\sup_{0 < y \leq x} \frac{B(y)}{V(y)} \right) \right]^r w(x) dx \right)^{1/r}, \\ D_3 &:= \left(\int_0^\infty \left(\int_x^\infty \left[\sup_{t \leq \tau < \infty} \frac{u(\tau)}{V^2(\tau)} \right]^q w(t) dt \right)^{r/p} \left[\sup_{x \leq \tau < \infty} \frac{u(\tau)}{V^2(\tau)} \right]^q V^r(x) w(x) dx \right)^{1/r}, \\ D_4 &:= \left(\int_0^\infty W^{r/p}(x) \left[\sup_{x \leq \tau < \infty} \left[\sup_{\tau \leq y < \infty} \frac{u(y)}{V^2(y)} \right] V(x) \right]^r w(x) dx \right)^{1/r}; \end{aligned}$$

(v) $p \leq q$, and in this case $c \approx E_1 + E_2$, where

$$\begin{aligned} E_1 &:= \sup_{x>0} \left(\left[\sup_{x \leq \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q \int_0^x w(t) dt + \int_x^\infty \left[\sup_{t \leq \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q w(t) dt \right)^{1/q} \sup_{0 < y \leq x} \frac{B(y)}{V^{1/p}(y)}, \\ E_2 &:= \sup_{x>0} \left(\left[\sup_{x \leq y < \infty} \frac{u^p(y)}{V^2(y)} \right]^{q/p} \int_0^x w(t) dt + \int_x^\infty \left[\sup_{t \leq y < \infty} \frac{u^p(y)}{V^2(y)} \right]^{q/p} w(t) dt \right)^{1/q} V^{1/p}(x); \end{aligned}$$

(vi) $q < p$, and in this case $c \approx F_1 + F_2 + F_3 + F_4$, where

$$\begin{aligned} F_1 &:= \left(\int_0^\infty \left(\int_0^x w(t) dt \right)^{r/p} \left[\sup_{x \leq \tau < \infty} \left[\sup_{\tau \leq y < \infty} \frac{u(y)}{B(y)} \right]^p \left(\sup_{0 < y \leq \tau} \frac{B(y)^p}{V(y)} \right) \right]^{r/p} w(x) dx \right)^{1/r}, \\ F_2 &:= \left(\int_0^\infty \left(\int_x^\infty \left[\sup_{t \leq \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q w(t) dt \right)^{r/p} \left[\sup_{0 < \tau \leq x} \frac{B^p(\tau)}{V(\tau)} \right]^{r/p} \left[\sup_{x \leq \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q w(x) dx \right)^{1/r}, \\ F_3 &:= \left(\int_0^\infty \left(\int_0^x w(t) dt \right)^{r/p} \left(\sup_{x \leq \tau < \infty} \left[\sup_{\tau \leq y < \infty} \frac{u^p(y)}{V^2(y)} \right] V(\tau) \right)^{r/p} w(x) dx \right)^{1/r}, \\ F_4 &:= \left(\int_0^\infty \left(\int_x^\infty \left[\sup_{t \leq y < \infty} \frac{u^p(y)}{V^2(y)} \right]^{q/p} w(t) dt \right)^{r/p} \left[\sup_{x \leq y < \infty} \frac{u^p(y)}{V^2(y)} \right]^{q/p} V^{r/p}(x) w(x) dx \right)^{1/r}. \end{aligned}$$

Now we give the solution of inequality (1.9), when $p = \infty$ or $q = \infty$.

Theorem 2.7. Let $0 < p < \infty$. Assume that $b \in \mathcal{W}(0, \infty)$, $u, w \in \mathcal{W}(0, \infty) \cap C(0, \infty)$ is such that $0 < B(t) < \infty$, $0 < V(x) < \infty$ and $0 < W(x) < \infty$ for all $x > 0$. Then inequality

$$(2.5) \quad \|T_{u,b}f\|_{\infty,w,(0,\infty)} \leq c \|f\|_{p,v,(0,\infty)}, \quad f \in \mathfrak{M}^\downarrow(0, \infty)$$

is satisfied with the best constant c if and only if:

(i) $1 < p$, and in this case $c \approx G_1 + G_2$, where

$$\begin{aligned} G_1 &:= \sup_{x>0} \left(\sup_{x \leq t < \infty} \left[\sup_{0 < \tau \leq t} w(\tau) \right] \frac{u(t)}{B(t)} \right) \left(\int_0^x \left(\frac{B(y)}{V(y)} \right)^{p'} v(y) dy \right)^{1/p'}, \\ G_2 &:= \sup_{x>0} \left(\sup_{x \leq t < \infty} \left[\sup_{0 < \tau \leq t} w(\tau) \right] \frac{u(t)}{V^2(t)} \right) \left(\int_0^x V^{p'}(y) v(y) dy \right)^{1/p'}; \end{aligned}$$

(ii) $p \leq 1$, and in this case $c \approx H_1 + H_2$, where

$$\begin{aligned} H_1 &:= \sup_{x>0} \left(\sup_{0 < y \leq x} \left(B(y) \sup_{y \leq t < \infty} \left[\sup_{0 < \tau \leq t} w(\tau) \right] \frac{u(t)}{B(t)} \right) \right) V^{-1/p}(x), \\ H_2 &:= \sup_{x>0} \left(\sup_{x \leq t < \infty} \left[\sup_{0 < \tau \leq t} w(\tau) \right] \frac{u(t)}{B(t)} \right) \frac{B(x)}{V^{1/p}(x)}. \end{aligned}$$

Proof. Whenever F, G are non-negative measurable functions on $(0, \infty)$ and F is non-increasing, then

$$\operatorname{ess\,sup}_{t \in (0, \infty)} F(t)G(t) = \operatorname{ess\,sup}_{t \in (0, \infty)} F(t) \operatorname{ess\,sup}_{\tau \in (0, t)} G(\tau);$$

likewise, when F is non-decreasing, then

$$\operatorname{ess\,sup}_{t \in (0, \infty)} F(t)G(t) = \operatorname{ess\,sup}_{t \in (0, \infty)} F(t) \operatorname{ess\,sup}_{\tau \in (t, \infty)} G(\tau).$$

Hence

$$(2.6) \quad \|T_{u,b}f\|_{\infty, w, (0, \infty)} = \sup_{x>0} \left(\sup_{x \leq t < \infty} \left[\sup_{0 < \tau \leq t} w(\tau) \right] \frac{u(t)}{B(t)} \right) \int_0^x f(y)b(y)dy,$$

and inequality (2.5) is equivalent to the inequality

$$(2.7) \quad \sup_{x>0} \left(\sup_{x \leq t < \infty} \left[\sup_{0 < \tau \leq t} w(\tau) \right] \frac{u(t)}{B(t)} \right) \int_0^x f(y)b(y)dy \leq c \left(\int_0^\infty f^p(y)v(y)dy \right)^{1/p}, \quad f \in \mathfrak{M}^\downarrow(0, \infty).$$

(i) Let $p > 1$. As in the proof of [28, Theorem 5.1] it can be shown that (2.7) is equivalent to the following two inequalities:

$$\begin{aligned} \sup_{x>0} \left[\sup_{0 < \tau \leq x} w(\tau) \right] \frac{u(x)}{B(x)} \int_0^x h(y)dy &\leq c \left(\int_0^\infty h^p(y) \left(\frac{V(y)}{B(y)} \right)^p v^{1-p}(y)dy \right)^{1/p}, \quad h \in \mathfrak{M}^+(0, \infty), \\ \sup_{x>0} \left[\sup_{0 < \tau \leq x} w(\tau) \right] \frac{u(x)}{V^2(x)} \int_0^x h(y)dy &\leq c \left(\int_0^\infty \left(\frac{h(y)}{V(y)} \right)^p v^{1-p}(y)dy \right)^{1/p}, \quad h \in \mathfrak{M}^+(0, \infty), \end{aligned}$$

which hold if and only if $G_1 < \infty$ and $G_2 < \infty$, respectively (see, for instance, [42, 43, 51]).

(ii) Let $p \leq 1$. It is known that inequality (2.7) holds if and only if

$$\sup_{x>0} \left(\sup_{y>0} B(\min\{x, y\}) \left(\sup_{y \leq t < \infty} \left[\sup_{0 < \tau \leq t} w(\tau) \right] \frac{u(t)}{B(t)} \right) \right) V^{-1/p}(x) < \infty$$

(see, for instance, [27, Theorem 5.1, (v)]), which is evidently holds iff $H_1 < \infty$ and $H_2 < \infty$. □

Theorem 2.8. Assume that $b \in \mathcal{W}(0, \infty)$, $u, w \in \mathcal{W}(0, \infty) \cap C(0, \infty)$ is such that $0 < B(t) < \infty$, $0 < V(x) < \infty$ and $0 < W(x) < \infty$ for all $x > 0$. Then inequality

$$(2.8) \quad \|T_{u,b}f\|_{\infty, w, (0, \infty)} \leq c \|f\|_{\infty, v, (0, \infty)}, \quad f \in \mathfrak{M}^\downarrow(0, \infty)$$

holds if and only if

$$I := \sup_{x>0} \left(\int_0^x \frac{b(y)dy}{\operatorname{ess\,sup}_{\tau \in (0, y)} v(\tau)} \right) \left[\sup_{0 < \tau \leq x} w(\tau) \right] \frac{u(x)}{B(x)} < \infty.$$

Moreover, the best constant c in (2.8) satisfies $c \approx I$.

Proof. By (2.6), we know that inequality (2.8) is equivalent to the inequality

$$(2.9) \quad \sup_{x>0} \left(\sup_{x \leq t < \infty} \left[\sup_{0 < \tau \leq t} w(\tau) \right] \frac{u(t)}{B(t)} \right) \int_0^x f(y)b(y)dy \leq c \operatorname{ess\,sup}_{x>0} f(x)v(x), \quad f \in \mathfrak{M}^\downarrow(0, \infty),$$

which holds if and only if

$$\sup_{x>0} \left(\int_0^x \frac{b(y)dy}{\operatorname{ess\,sup}_{\tau \in (0, y)} v(\tau)} \right) \left[\sup_{0 < \tau \leq x} w(\tau) \right] \frac{u(x)}{B(x)} < \infty$$

(see, for instance, [27, Theorem 5.1, (viii)]). □

3. MAIN RESULTS

In this section we give statements and proofs of our main results.

Lemma 3.1. *Let $0 < r < \infty$. Assume that $\phi \in Q_r$. Suppose that X is a quasi-Banach function space on a measure space (\mathbb{R}^n, dx) . Moreover, assume that X satisfy a lower r -estimate. Then there exists $C > 0$ such that for any function f from X and any finite pairwise disjoint collection cubes $\{Q_j\}$ on \mathbb{R}^n*

$$(3.1) \quad \min_i \frac{\|f\chi_{Q_i}\|_X}{\phi(|Q_i|)} \leq C \frac{\|f\chi_{\cup_i Q_i}\|_X}{\phi(|\cup_i Q_i|)}$$

holds true.

Proof. Denote by

$$A := \min_i \frac{\|f\chi_{Q_i}\|_X}{\phi(|Q_i|)}.$$

Since $\phi \in Q_r$, we have that

$$A\phi(|\cup_i Q_i|) = A\phi\left(\sum_i |Q_i|\right) \lesssim A\left(\sum_i \phi(|Q_i|)^r\right)^{1/r} \leq \left(\sum_i \|f\chi_{Q_i}\|_X^r\right)^{1/r}.$$

On using the r -lower estimate property of X , we get that

$$A\phi(|\cup_i Q_i|) \lesssim \left\| \sum_{i=1}^n f\chi_{Q_i} \right\|_X = \|f\chi_{\cup_i Q_i}\|_X.$$

□

Lemma 3.2. *Let $0 < r < \infty$. Assume that $\phi \in Q_r$. Suppose that X is a quasi-Banach function space satisfying a lower r -estimate. Then, for any $t > 0$,*

$$(3.2) \quad (M_{\phi,X}f)^*(t) \leq C \sup_{|E|>t/3^n} \frac{\|f\chi_E\|_X}{\phi(|E|)},$$

where $C > 0$ is the constant appearing in (3.1).

Proof. The statement follows by Theorem 2.2 and Lemma 3.1. □

Lemma 3.3. *Let $0 < r < \infty$. Assume that $\phi \in Q_r$. Suppose that X is a r.i. quasi-Banach function space satisfying a lower r -estimate. Then, for any $t > 0$,*

$$(3.3) \quad (M_{\phi,X}f)^*(t) \leq C \sup_{\tau>t} \frac{\|f^*\chi_{[0,\tau]}\|_{\bar{X}}}{\phi(\tau)},$$

where $C > 0$ is constant independent of f and t .

Proof. By Lemma 3.2, we have that

$$(M_{\phi,X}f)^*(t) \leq C \sup_{|E|>t/3^n} \frac{\|f\chi_E\|_X}{\phi(|E|)} = C \sup_{|E|>t/3^n} \frac{\|(f\chi_E)^*\|_{\bar{X}}}{\phi(|E|)} \leq C \sup_{|E|>t/3^n} \frac{\|f^*\chi_{[0,|E|]}\|_{\bar{X}}}{\phi(|E|)} \leq C \sup_{\tau>t} \frac{\|f^*\chi_{[0,\tau]}\|_{\bar{X}}}{\phi(\tau)}.$$

□

Corollary 3.4. *Let $0 < \alpha \leq r < \infty$, $\phi \in Q_r$ and $b \in \mathcal{W}(0, \infty)$ be such that $B(\infty) = \infty$, $B \in \Delta_2$ and $B(t)/t^{\alpha/r}$ is quasi-increasing. Then there exists a constant $C > 0$ such that for any measurable function f on \mathbb{R}^n the inequality*

$$(M_{\phi,\Lambda^\alpha(b)}f)^*(t) \leq C \sup_{\tau>t} \frac{\left(\int_0^\tau (f^*)^\alpha(y) b(y) dy \right)^{1/\alpha}}{\phi(\tau)}$$

holds.

Proof. In view of Theorem 2.4, $\Lambda^\alpha(b)$ satisfies a lower r -estimate. Then the statement follows from Lemma 3.3, when $X = \Lambda^\alpha(b)$. \square

Corollary 3.5. *Let $0 < q \leq p < \infty$. Then there exists a constant $C > 0$ such that for any measurable function f on \mathbb{R}^n the inequality*

$$(M_{p,q}f)^*(t) \leq \frac{C}{t^{1/p}} \left(\int_0^t (f^*)^q(y) y^{q/p-1} dy \right)^{1/q}$$

holds.

Proof. Let $\alpha = q$, $b(t) = t^{q/p-1}$ and $\phi(t) = t^{1/p}$ ($t > 0$). Then $M_{p,q} = M_{\phi,\Lambda^\alpha(b)}$. It is clear that $B(t) \approx t^{q/p}$ ($t > 0$). Since $\phi \in Q_r$, $B \in \Delta_2$ and $B(t)/t^{q/r}$ is quasi-increasing when $r = p \geq q$, by Corollary 3.4, we get that

$$(M_{p,q}f)^*(t) \leq C \sup_{\tau > t} \frac{1}{\tau^{1/p}} \left(\int_0^\tau (f^*)^q(y) y^{q/p-1} dy \right)^{1/q} = \frac{C}{t^{1/p}} \left(\int_0^t (f^*)^q(y) y^{q/p-1} dy \right)^{1/q}.$$

\square

Corollary 3.6. *Let $s \in (0, \infty)$, $\gamma \in (0, n)$ and $\mathbb{A} = (A_0, A_\infty) \in \mathbb{R}^2$. Then there exists a constant $C > 0$ depending only in n, s, γ and \mathbb{A} such that for all $f \in \mathfrak{M}(\mathbb{R}^n)$ and every $t \in (0, \infty)$*

$$(3.4) \quad (M_{s,\gamma,\mathbb{A}}f)^*(t) \leq C \left[\sup_{\tau > t} \tau^{\gamma/n-1} \ell^{-s\mathbb{A}}(\tau) \int_0^\tau (f^*)^s(y) dy \right]^{1/s}.$$

Proof. It is mentioned in the introduction that $M_{\phi,\Lambda^\alpha(b)} \approx M_{s,\gamma,\mathbb{A}}$, when $\alpha = s$, $b \equiv 1$ and $\phi(t) = t^{(n-\gamma)/(sn)} \ell^{\mathbb{A}}(t)$, ($t > 0$). Let $r = s$. Writing $\phi = \omega \cdot g$, where $\omega(t) = t^{1/s}$ and $g(t) = t^{-\gamma/(sn)} \ell^{\mathbb{A}}(t)$, ($t > 0$), observing that $\omega \in Q_s$ and g is quasi-decreasing, we claim that $\phi \in Q_r$. On the other side, since $B(t) = t$, $t > 0$, we get that $B \in \Delta_2$ and $B(t)/t^{\alpha/r} \equiv 1$ is quasi-increasing. Hence, by Corollary 3.4, inequality (3.4) holds. \square

Lemma 3.7. *Let $0 < r < \infty$. Assume that $\phi \in \Delta_2$ is a quasi-increasing function on $(0, \infty)$. Suppose that X is a r.i. quasi-Banach function space. Then, for any $t > 0$,*

$$(3.5) \quad (M_{\phi,X}f)^*(t) \geq c \sup_{\tau > t} \frac{\|f^* \chi_{[0,\tau]}\|_{\bar{X}}}{\phi(\tau)}, \quad f \in \mathfrak{M}^{\text{rad},\downarrow}(\mathbb{R}^n),$$

where $c > 0$ is constant independent of f and t .

Proof. Let f be any function from $\mathfrak{M}^{\text{rad},\downarrow}$. For every $x, y \in \mathbb{R}^n$ such that $|y| > |x|$, we have that

$$(M_{\phi,X}f)(x) \gtrsim \frac{\|f \chi_{B(0,|y|)}\|_X}{\phi(|B(0,|y|)|)}.$$

Since $(f \chi_{B(0,|y|)})^*(t) = f^*(t) \chi_{[0,|B(0,|y|)|)}(t)$, $t > 0$, we get that

$$(M_{\phi,X}f)(x) \gtrsim \frac{\|f^* \chi_{[0,|B(0,|y|)|)}\|_{\bar{X}}}{\phi(|B(0,|y|)|)}.$$

Hence

$$(M_{\phi,X}f)(x) \gtrsim \sup_{|y| > |x|} \frac{\|f^* \chi_{[0,|B(0,|y|)|)}\|_{\bar{X}}}{\phi(|B(0,|y|)|)} = \sup_{|y| > |x|} \frac{\|f^* \chi_{[0,\omega_n|y|^n]}\|_{\bar{X}}}{\phi(\omega_n|y|^n)} = \sup_{\tau > \omega_n|x|^n} \frac{\|f^* \chi_{[0,\tau]}\|_{\bar{X}}}{\phi(\tau)},$$

where ω_n is the Lebesgue measure of the unit ball in \mathbb{R}^n .

Recall that

$$f^*(t) = \sup_{|E|=t} \operatorname{ess\,inf}_{x \in E} |f(x)|, \quad t \in (0, \infty),$$

(see, for instance, [11, p. 33]).

On taking rearrangements, we obtain that

$$\begin{aligned}
(M_{\phi,X}f)^*(t) &= \sup_{|E|=t} \operatorname{ess\,inf}_{x \in E} (M_{\phi,X}f)(x) \\
&\geq \operatorname{ess\,inf}_{x \in B(0,(t/\omega_n)^{1/n})} (M_{\phi,X}f)(x) \\
&\gtrsim \operatorname{ess\,inf}_{x \in B(0,(t/\omega_n)^{1/n})} \sup_{\tau > \omega_n |x|^n} \frac{\|f^* \chi_{[0,\tau]}\|_{\bar{X}}}{\phi(\tau)} \\
&= \operatorname{ess\,inf}_{0 \leq s < t} \sup_{\tau > s} \frac{\|f^* \chi_{[0,\tau]}\|_{\bar{X}}}{\phi(\tau)} \\
&= \sup_{\tau > t} \frac{\|f^* \chi_{[0,\tau]}\|_{\bar{X}}}{\phi(\tau)}.
\end{aligned}$$

□

Combining Lemmas 3.3 and 3.7, we get the following statement.

Theorem 3.8. *Let $0 < p, q < \infty$, $0 < r < \infty$. Assume that $\phi \in Q_r$ is a quasi-increasing function on $(0, \infty)$. Suppose that X is a r.i. quasi-Banach function space satisfying a lower r -estimate. Then:*

(a) $M_{\phi,X}$ is bounded from $\Lambda^p(v)$ to $\Lambda^q(w)$, that is, the inequality

$$\|M_{\phi,X}f\|_{\Lambda^q(w)} \leq C\|f\|_{\Lambda^p(v)}$$

holds for all $f \in \mathfrak{M}(\mathbb{R}^n)$ if and only if the inequality

$$\left(\int_0^\infty \left[\sup_{\tau > t} \frac{\|\psi \chi_{[0,\tau]}\|_{\bar{X}}}{\phi(\tau)} \right]^q w(t) dt \right)^{1/q} \leq C \left(\int_0^\infty (\psi(t))^p v(t) dt \right)^{1/p}$$

holds for all $\psi \in \mathfrak{M}^+((0, \infty); \downarrow)$.

(b) $M_{\phi,X}$ is bounded from $\Lambda^p(v)$ to $\Lambda^{q,\infty}(w)$, that is, the inequality

$$\|M_{\phi,X}f\|_{\Lambda^{q,\infty}(w)} \leq C\|f\|_{\Lambda^p(v)}$$

holds for all $f \in \mathfrak{M}(\mathbb{R}^n)$ if and only if the inequality

$$\sup_{t>0} (W(t))^{1/q} \sup_{\tau > t} \frac{\|\psi \chi_{[0,\tau]}\|_{\bar{X}}}{\phi(\tau)} \leq C \left(\int_0^\infty (\psi(t))^p v(t) dt \right)^{1/p}$$

holds for all $\psi \in \mathfrak{M}^+((0, \infty); \downarrow)$.

(c) $M_{\phi,X}$ is bounded from $\Lambda^{p,\infty}(v)$ to $\Lambda^{q,\infty}(w)$, that is, the inequality

$$\|M_{\phi,X}f\|_{\Lambda^{q,\infty}(w)} \leq C\|f\|_{\Lambda^{p,\infty}(v)}$$

holds for all $f \in \mathfrak{M}(\mathbb{R}^n)$ if and only if the inequality

$$\sup_{t>0} (W(t))^{1/q} \sup_{\tau > t} \frac{\|\psi \chi_{[0,\tau]}\|_{\bar{X}}}{\phi(\tau)} \leq C \sup_{t>0} (V(t))^{1/q} \psi(t)$$

holds for all $\psi \in \mathfrak{M}^+((0, \infty); \downarrow)$.

3.1. Boundedness of $M_{\phi,\Lambda^\alpha(b)} : \Lambda^p(v) \rightarrow \Lambda^q(w)$, $0 < p, q < \infty$.

Theorem 3.9. *Let $0 < p, q < \infty$, $0 < \alpha \leq r < \infty$ and $v, w \in \mathcal{W}(0, \infty)$. Assume that $\phi \in Q_r$ is a quasi-increasing function. Moreover, assume that $b \in \mathcal{W}(0, \infty)$ is such that $0 < B(t) < \infty$ for all $x > 0$, $B(\infty) = \infty$, $B \in \Delta_2$ and $B(t)/t^{\alpha/r}$ is quasi-increasing. Then $M_{\phi,\Lambda^\alpha(b)}$ is bounded from $\Lambda^p(v)$ to $\Lambda^q(w)$, that is, the inequality*

$$\|M_{\phi,\Lambda^\alpha(b)}f\|_{\Lambda^q(w)} \leq C\|f\|_{\Lambda^p(v)}$$

holds for all $f \in \mathfrak{M}(\mathbb{R}^n)$ if and only if the inequality

$$(3.6) \quad \|T_{B/\phi^\alpha, b\psi}\|_{q/\alpha, w, (0, \infty)} \leq C^\alpha \|\psi\|_{p/\alpha, v, (0, \infty)}$$

holds for all $\psi \in \mathfrak{M}^+((0, \infty); \downarrow)$.

Proof. The statement follows from Theorem 3.8, (a), when $X = \Lambda^\alpha(b)$. \square

Theorem 3.10. Let $0 < p, q < \infty$, $0 < \alpha \leq r < \infty$ and $v, w \in \mathcal{W}(0, \infty)$. Assume that $\phi \in \mathcal{Q}_r$ is a quasi-increasing function. Moreover, assume that $b \in \mathcal{W}(0, \infty)$ is such that $0 < B(t) < \infty$ for all $x > 0$, $B(\infty) = \infty$, $B \in \Delta_2$ and $B(t)/t^{\alpha/r}$ is quasi-increasing. Then $M_{\phi, \Lambda^\alpha(b)}$ is bounded from $\Lambda^p(v)$ to $\Lambda^q(w)$ if and only if:

(i) $\alpha < p \leq q$, and in this case $c \approx \mathcal{A}_1 + \mathcal{A}_2$, where

$$\mathcal{A}_1 := \sup_{x>0} \left(\phi^{-q}(x)W(x) + \int_x^\infty \phi^{-q}(t)w(t)dt \right)^{\frac{1}{q}} \left(\int_0^x \left(\frac{B(y)}{V(y)} \right)^{\frac{p}{p-\alpha}} v(y)dy \right)^{\frac{p-\alpha}{pq}},$$

$$\mathcal{A}_2 := \sup_{x>0} \left(\left[\sup_{x \leq \tau < \infty} \frac{B(\tau)}{\phi^\alpha(\tau)V^2(\tau)} \right]^{\frac{q}{\alpha}} W(x) + \int_x^\infty \left[\sup_{t \leq \tau < \infty} \frac{B(\tau)}{\phi^\alpha(\tau)V^2(\tau)} \right]^{\frac{q}{\alpha}} w(t)dt \right)^{\frac{1}{q}} \left(\int_0^x V^{\frac{p}{p-\alpha}} v \right)^{\frac{p-\alpha}{pq}};$$

(ii) $\alpha = p \leq q$, and in this case $c \approx \mathcal{B}_1 + \mathcal{B}_2$, where

$$\mathcal{B}_1 := \sup_{x>0} \left(\phi^{-q}(x)W(x) + \int_x^\infty \phi^{-q}(t)w(t)dt \right)^{1/q} \left(\sup_{0 < y \leq x} \frac{B(y)}{V(y)} \right)^{\frac{1}{\alpha}},$$

$$\mathcal{B}_2 := \sup_{x>0} \left(\left[\sup_{x \leq \tau < \infty} \frac{B(\tau)}{\phi^\alpha(\tau)V^2(\tau)} \right]^{\frac{q}{\alpha}} W(x) + \int_x^\infty \left[\sup_{t \leq \tau < \infty} \frac{B(\tau)}{\phi^\alpha(\tau)V^2(\tau)} \right]^{\frac{q}{\alpha}} w(t)dt \right)^{\frac{1}{q}} V^{\frac{1}{\alpha}}(x);$$

(iii) $\alpha < p$ and $q < p$, and in this case $c \approx \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4$, where

$$\mathcal{C}_1 := \left(\int_0^\infty \left(\int_x^\infty \phi^{-q}(t)w(t)dt \right)^{\frac{q}{p-q}} \phi^{-q}(x) \left(\int_0^x \left(\frac{B(y)}{V(y)} \right)^{\frac{p}{p-\alpha}} v(y)dy \right)^{\frac{q(p-\alpha)}{\alpha(p-q)}} w(x)dx \right)^{\frac{p-q}{pq}},$$

$$\mathcal{C}_2 := \left(\int_0^\infty W^{\frac{q}{p-q}}(x) \left[\sup_{x \leq \tau < \infty} \phi^{-\alpha}(\tau) \left(\int_0^\tau \left(\frac{B(y)}{V(y)} \right)^{\frac{p}{p-\alpha}} v(y)dy \right)^{\frac{p-\alpha}{p}} \right]^{\frac{pq}{\alpha(p-q)}} w(x)dx \right)^{\frac{p-q}{pq}},$$

$$\mathcal{C}_3 := \left(\int_0^\infty \left(\int_x^\infty \left[\sup_{t \leq \tau < \infty} \frac{B(\tau)}{\phi^\alpha(\tau)V^2(\tau)} \right]^{\frac{q}{\alpha}} w(t)dt \right)^{\frac{q}{p-q}} \left[\sup_{x \leq \tau < \infty} \frac{B(\tau)}{\phi^\alpha(\tau)V^2(\tau)} \right]^{\frac{q}{\alpha}} \left(\int_0^x V^{\frac{p}{p-\alpha}} v \right)^{\frac{q(p-\alpha)}{\alpha(p-q)}} w(x)dx \right)^{\frac{p-q}{pq}},$$

$$\mathcal{C}_4 := \left(\int_0^\infty W^{\frac{q}{p-q}}(x) \left[\sup_{x \leq \tau < \infty} \left[\sup_{\tau \leq y < \infty} \frac{B(y)}{\phi^\alpha(y)V^2(y)} \right] \left(\int_0^\tau V^{\frac{p}{p-\alpha}} v \right)^{\frac{p-\alpha}{p}} \right]^{\frac{pq}{\alpha(p-q)}} w(x)dx \right)^{\frac{p-q}{pq}};$$

(iv) $q < \alpha = p$, and in this case $c \approx \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4$, where

$$\mathcal{D}_1 := \left(\int_0^\infty \left(\int_x^\infty \phi^{-q}(t)w(t)dt \right)^{\frac{q}{p-q}} \phi^{-q}(x) \left(\sup_{0 < y \leq x} \frac{B(y)}{V(y)} \right)^{\frac{pq}{\alpha(p-q)}} w(x)dx \right)^{\frac{p-q}{pq}},$$

$$\mathcal{D}_2 := \left(\int_0^\infty W^{\frac{q}{p-q}}(x) \left[\sup_{x \leq \tau < \infty} \phi^{-\alpha}(\tau) \left(\sup_{0 < y \leq \tau} \frac{B(y)}{V(y)} \right) \right]^{\frac{pq}{\alpha(p-q)}} w(x)dx \right)^{\frac{p-q}{pq}},$$

$$\mathcal{D}_3 := \left(\int_0^\infty \left(\int_x^\infty \left[\sup_{t \leq \tau < \infty} \frac{B(\tau)}{\phi^\alpha(\tau)V^2(\tau)} \right]^{\frac{q}{\alpha}} w(t)dt \right)^{\frac{q}{p-q}} \left[\sup_{x \leq \tau < \infty} \frac{B(\tau)}{\phi^\alpha(\tau)V^2(\tau)} \right]^{\frac{q}{\alpha}} V^{\frac{pq}{\alpha(p-q)}}(x)w(x)dx \right)^{\frac{p-q}{pq}},$$

$$\mathcal{D}_4 := \left(\int_0^\infty W^{\frac{q}{p-q}}(x) \left[\sup_{x \leq \tau < \infty} \left[\sup_{\tau \leq y < \infty} \frac{B(y)}{\phi^\alpha(y)V^2(y)} \right] V(\tau) \right]^{\frac{pq}{\alpha(p-q)}} w(x)dx \right)^{\frac{p-q}{pq}};$$

(v) $p \leq \alpha$, $p \leq q$ and in this case $c \approx \mathcal{E}_1 + \mathcal{E}_2$, where

$$\mathcal{E}_1 := \sup_{x>0} \left(\phi^{-q}(x)W(x) + \int_x^\infty \phi^{-q}(t)w(t)dt \right)^{\frac{1}{q}} \sup_{0 < y \leq x} \frac{B^{\frac{1}{\alpha}}(y)}{V^{\frac{1}{p}}(y)},$$

$$\mathcal{E}_2 := \sup_{x>0} \left(\left[\sup_{x \leq y < \infty} \frac{B^{\frac{1}{\alpha}}(y)}{\phi(y)V^{\frac{2}{p}}(y)} \right]^q W(x) + \int_x^\infty \left[\sup_{t \leq y < \infty} \frac{B^{\frac{1}{\alpha}}(y)}{\phi(y)V^{\frac{2}{p}}(y)} \right]^q w(t) dt \right)^{\frac{1}{q}} V^{\frac{1}{p}}(x);$$

(vi) $p \leq \alpha$, $q < p$, and in this case $c \approx \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 + \mathcal{F}_4$, where

$$\begin{aligned} \mathcal{F}_1 &:= \left(\int_0^\infty W^{\frac{q}{p-q}}(x) \left[\sup_{x \leq \tau < \infty} \phi^{-q}(\tau) \left(\sup_{0 < y \leq \tau} \frac{B(y)}{V^{\frac{\alpha}{p}}(y)} \right)^{\frac{pq}{\alpha(p-q)}} w(x) dx \right]^{\frac{p-q}{pq}} \right)^{\frac{p-q}{pq}}, \\ \mathcal{F}_2 &:= \left(\int_0^\infty \left(\int_x^\infty \phi^{-q}(t) w(t) dt \right)^{\frac{q}{p-q}} \left[\sup_{0 < \tau \leq x} \frac{B(\tau)}{V^{\frac{\alpha}{p}}(\tau)} \right]^{\frac{pq}{\alpha(p-q)}} \phi^{-q}(x) w(x) dx \right)^{\frac{p-q}{pq}}, \\ \mathcal{F}_3 &:= \left(\int_0^\infty W^{\frac{q}{p-q}}(x) \left(\sup_{x \leq \tau < \infty} \left[\sup_{\tau \leq y < \infty} \frac{B^{\frac{1}{\alpha}}(y)}{\phi(y)V^{\frac{2}{p}}(y)} \right] V^{\frac{1}{p}}(\tau) \right)^{\frac{pq}{p-q}} w(x) dx \right)^{\frac{p-q}{pq}}, \\ \mathcal{F}_4 &:= \left(\int_0^\infty \left(\int_x^\infty \left[\sup_{t \leq y < \infty} \frac{B^{\frac{1}{\alpha}}(y)}{\phi(y)V^{\frac{2}{p}}(y)} \right]^q w(t) dt \right)^{\frac{q}{p-q}} \left[\sup_{x \leq y < \infty} \frac{B^{\frac{1}{\alpha}}(y)}{\phi(y)V^{\frac{2}{p}}(y)} \right]^q V^{\frac{q}{p-q}}(x) w(x) dx \right)^{\frac{p-q}{pq}}. \end{aligned}$$

Proof. The statement follows from Theorems 3.9 and 2.6. \square

3.2. Boundedness of $M_{\phi, \Lambda^\alpha(b)} : \Lambda^p(v) \rightarrow \Lambda^{q, \infty}(w)$, $0 < p, q < \infty$.

Theorem 3.11. *Let $0 < p, q < \infty$, $0 < \alpha \leq r < \infty$ and $v, w \in \mathcal{W}(0, \infty)$. Assume that $\phi \in \mathcal{Q}_r$ is a quasi-increasing function. Moreover, assume that $b \in \mathcal{W}(0, \infty)$ is such that $0 < B(t) < \infty$ for all $x > 0$, $B(\infty) = \infty$, $B \in \Delta_2$ and $B(t)/t^{\alpha/r}$ is quasi-increasing. Then $M_{\phi, \Lambda^\alpha(b)}$ is bounded from $\Lambda^p(v)$ to $\Lambda^{q, \infty}(w)$, that is, the inequality*

$$\|M_{\phi, \Lambda^\alpha(b)}\|_{\Lambda^{q, \infty}(w)} \leq C \|f\|_{\Lambda^p(v)}$$

holds for all $f \in \mathfrak{M}(\mathbb{R}^n)$ if and only if the inequality

$$(3.7) \quad \|T_{B/\phi^\alpha, b}\psi\|_{\infty, W^{\alpha/q}, (0, \infty)} \leq C^\alpha \|\psi\|_{p/\alpha, v, (0, \infty)}$$

holds for all $\psi \in \mathfrak{M}^+((0, \infty); \downarrow)$.

Proof. The statement follows from Theorem 3.8, (b), when $X = \Lambda^\alpha(b)$. \square

Theorem 3.12. *Let $0 < p, q < \infty$, $0 < \alpha \leq r < \infty$ and $v, w \in \mathcal{W}(0, \infty)$. Assume that $\phi \in \mathcal{Q}_r$ is a quasi-increasing function. Moreover, assume that $b \in \mathcal{W}(0, \infty)$ is such that $0 < B(t) < \infty$ for all $x > 0$, $B(\infty) = \infty$, $B \in \Delta_2$ and $B(t)/t^{\alpha/r}$ is quasi-increasing. Then $M_{\phi, \Lambda^\alpha(b)}$ is bounded from $\Lambda^p(v)$ to $\Lambda^{q, \infty}(w)$ if and only if:*

(i) $\alpha \leq p$, and in this case $c \approx \mathcal{G}_1 + \mathcal{G}_2$, where

$$\begin{aligned} \mathcal{G}_1 &:= \sup_{x>0} \left[\sup_{x \leq t < \infty} \frac{W^{\frac{1}{q}}(t)}{\phi(t)} \right] \left(\int_0^x \left(\frac{B(y)}{V(y)} \right)^{\frac{p}{p-\alpha}} v(y) dy \right)^{\frac{p-\alpha}{p\alpha}}, \\ \mathcal{G}_2 &:= \sup_{x>0} \left[\sup_{x \leq t < \infty} \frac{W^{\frac{1}{q}}(t) B^{\frac{1}{\alpha}}(t)}{\phi(t) V^{\frac{2}{\alpha}}(t)} \right] \left(\int_0^x V^{\frac{p}{p-\alpha}} v \right)^{\frac{p-\alpha}{p\alpha}}; \end{aligned}$$

(ii) $p < \alpha$, and in this case $c \approx \mathcal{H}_1 + \mathcal{H}_2$, where

$$\begin{aligned} \mathcal{H}_1 &:= \sup_{x>0} \left(\sup_{0 < y \leq x} B^{\frac{1}{\alpha}}(y) \left[\sup_{y \leq t < \infty} \frac{W^{\frac{1}{q}}(t)}{\phi(t)} \right] \right) V^{-\frac{1}{p}}(x), \\ \mathcal{H}_2 &:= \sup_{x>0} \left[\sup_{x \leq t < \infty} \frac{W^{\frac{1}{q}}(t)}{\phi(t)} \right] \frac{B^{\frac{1}{\alpha}}(x)}{V^{\frac{1}{p}}(x)}. \end{aligned}$$

Proof. The statement follows by Theorems 3.11 and 2.7. \square

3.3. Boundedness of $M_{\phi, \Lambda^\alpha(b)} : \Lambda^{p, \infty}(v) \rightarrow \Lambda^{q, \infty}(w)$, $0 < p, q < \infty$.

Theorem 3.13. *Let $0 < p, q < \infty$, $0 < \alpha \leq r < \infty$ and $v, w \in \mathcal{W}(0, \infty)$. Assume that $\phi \in \mathcal{Q}_r$ is a quasi-increasing function. Moreover, assume that $b \in \mathcal{W}(0, \infty)$ is such that $0 < B(t) < \infty$ for all $x > 0$, $B(\infty) = \infty$, $B \in \Delta_2$ and $B(t)/t^{\alpha/r}$ is quasi-increasing. Then $M_{\phi, \Lambda^\alpha(b)}$ is bounded from $\Lambda^p(v)$ to $\Lambda^{q, \infty}(w)$, that is, the inequality*

$$\|M_{\phi, \Lambda^\alpha(b)}\|_{\Lambda^{q, \infty}(w)} \leq C \|f\|_{\Lambda^{p, \infty}(v)}$$

holds for all $f \in \mathfrak{M}(\mathbb{R}^n)$ if and only if the inequality

$$\|T_{B/\phi^\alpha, b}\psi\|_{\infty, W^{\alpha/q}, (0, \infty)} \leq C^\alpha \|\psi\|_{\infty, V^{\alpha/p}, (0, \infty)}$$

holds for all $\psi \in \mathfrak{M}^+((0, \infty); \downarrow)$.

Proof. The statement follows from Theorem 3.8, (c), when $X = \Lambda^\alpha(b)$. □

Theorem 3.14. *Let $0 < p, q < \infty$, $0 < \alpha \leq r < \infty$ and $v, w \in \mathcal{W}(0, \infty)$. Assume that $\phi \in \mathcal{Q}_r$ is a quasi-increasing function. Moreover, assume that $b \in \mathcal{W}(0, \infty)$ is such that $0 < B(t) < \infty$ for all $x > 0$, $B(\infty) = \infty$, $B \in \Delta_2$ and $B(t)/t^{\alpha/r}$ is quasi-increasing. Then $M_{\phi, \Lambda^\alpha(b)}$ is bounded from $\Lambda^{p, \infty}(v)$ to $\Lambda^{q, \infty}(w)$ if and only if*

$$\mathcal{I} := \sup_{x>0} \left(\int_0^x \frac{b(y)}{V^{\frac{\alpha}{p}}(y)} dy \right)^{\frac{1}{\alpha}} \frac{W^{\frac{1}{q}}(x)}{\phi(x)} < \infty.$$

Moreover, the best constant c in satisfies $c \approx \mathcal{I}$.

Proof. The statement follows by Theorems 3.13 and 2.8. □

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